General Formulation for the Aeroelastic Divergence of Composite Swept-Forward Wing Structures

L. Librescu*
Virginia Polytechnic Institute and State University, Blacksburg, Virginia
and
J. Simovich†
Tel-Aviv University, Tel-Aviv, Israel

This paper formulates a simple algorithm that allows for the determination, in a closed form, of the divergence instability of advanced composite swept (back and forward) wing structures. The warping restraint effect is incorporated into the analysis and its influence on the associated static instability condition is put into evidence. In this sense, it is shown that, in contrast to the case of conventional metallic wings where the warping restraint effect has a stabilizing influence only (which is strongly manifested in the case of small-aspect-ratio wings and declines for moderate AR wings), its influence becomes more complex in the case of anisotropic composite wing structures. The principal goal of this study is the divergence behavior of swept composite-type wings to the warping restraint effect being included in the analysis. The numerical examples illustrate the complex role played by the warping restraint effect on the divergence instability of composite wings.

Nomenclature

a_0 = lift curve slope
a_1, a_2 = two speed parameters, Eq. (8)
AR = wing aspect ratio
c = wing chord measured perpendicular to the reference axis
C_{ij} = stetching rigidities of the composite wing, Eq. (A5a)
D_{ij} = bending rigidities of the composite wing, Eq. (A5c)
$\tilde{\mathfrak{D}}_{ij}$ = composite coupling stiffness rigidities, Eq. (A16)
e = distance between the spanwise reference axis and aerodynamic centerline
f = normalized torsional deflection function, $f \equiv f(\eta)$, $f(1) = 1$
G = composite nondimensional torsional coupling parameter, Eq. (6b)
K = composite nondimensional bending coupling parameter, Eq. (6a)
K_{ij} = bending-stretching coupling rigidities, Eq. (A5b)
ℓ = wing semispan measured along the reference axis
L = aerodynamic lift force, Eq. (1a)
L_{ij} = coefficients defined by Eqs. (11)
q = dynamic pressure, $\equiv \frac{1}{2}\rho V^2$
S = warping rigidity, Eq. (7)
T = aerodynamic torsional moment, Eq. (1b)
U = strain energy of the three-dimensional body, Eqs. (A6) and (A7)
V = freestream velocity (parallel to the longitudinal axis of the airplane)
W = normalized bending deflection function, $W(1) = 1$, Eq. (9b)
y = distance along reference axis measured from the wing root
$Z, \tilde{Z}$ = bending deflection of the reference axis
η = y/ℓ
θ = torsional deflection of wing sections about the reference axis
$\theta_{(j)}$ = orthotropicity angle of each lamina j with respect to a rearward normal to the reference axis
Λ = sweep angle of the reference axis (positive for swept back)
ξ = $1-(1-\sigma)\eta$
σ = taper ratio, $\equiv c_T/c_R$

Subscripts and Superscripts

D = divergence quantities
R = wing root quantities
n = component normal to the leading edge
$(\ )_{,1}$ = $\partial(\ )/\partial\eta$
$(\ )'$ = $\partial(\ )/\partial y$
$(\ )^*$ = divergence quantity determined by including warping restraint effect

Introduction

THE expanded utilization of laminated anisotropic composites in the modern aircraft industry has led, during the last few years, to the development of a new concept in the design of aeronautical and space structures. Known as aeroelastic tailoring, this concept is defined as the technology applied to flight vehicle structures (and to other devices experiencing aeroelastic instability phenomena as, e.g., turbine and helicopter blades) in which framework the exotic characteristics of advanced filamentary composite materials are properly used in order to enhance their aeroelastic response characteristics. One of the best and most efficient applications of this concept is constituted by the forward-swept wing (FSW) aircraft. As is well known, in the case of swept-back wings, the bending deformations tend to reduce the local angle of attack (and implicitly the aerodynamic load), while in the case of a swept-forward wing, the bending deformations have an opposite effect, tending to increase the local angle of attack.

The above-mentioned effects, referred to as wash-out and wash-in, respectively, result either in an increase of the aeroelastic divergence speed or, in the second instance, in a

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*Professor, Department of Engineering Science and Mechanics (on leave from Faculty of Engineering, Tel-Aviv University, Tel-Aviv, Israel).

†Graduate Student, Faculty of Engineering.

low static instability speed. In the fundamental works[1,2] devoted to the study of the divergence instability of swept metallic wings, it was shown that this behavior practically precludes the consideration of FSW aircraft as a possible option, in spite of its potential advantages in aerodynamics and performance. In an attempt to alleviate this instability phenomenon jeopardizing the free employment of FSW aircraft, the studies prompted by Krone,[3] continued by Weisshaar,[4-7] and followed by a series of other theoretical and experimental research works[8-13] have revealed that a composite forward-swept wing can be aeroelastically tailored to overcome this adverse static instability phenomenon. (The aeroelastic tailoring concept has also been applied to supersonic panel flutter problems. For an extensive review of the state-of-the-art of this topic, see Chap. 1 of Ref. 14.)

Concerning the general framework in which this problem was modeled and the character of the solutions obtained, it should be pointed out that: 1) the concept of a simple anisotropic composite plate-beam model originated in the studies[4-7,15] was used throughout these investigations and 2) the divergence instability solutions for composite swept wings encompass both closed-form[4,6-8] and numerical[10,13] solutions. With regard to the closed-form solutions, they are either exact[4-6] or approximate[8] ones. (Here the term *exact* is used in the sense that the divergence instability solution was obtained in the frame work of some assumptions *initially stipulated.* This means that no other assumptions beyond the initial ones were used in the treatment of the problem.) It should also be mentioned that, as far as the authors of this paper are aware, with the exception of Ref. 13, the free warping assumption for wing twist has been unanimously adopted in the treatment of this problem. However, the results obtained in Refs. 13 and 16 related to the divergence instability, as well as the ones derived in Refs. 17-21, reveal the great importance of the axial warping restraint effect (WRE) on the behavior of cantilevered-type structures. The goal of this paper is twofold: 1) to develop a simple algorithm allowing, in an explicit form, the determination of the static instability conditions for swept composite-type wings with the warping restraint effect being incorporated, and 2) to elucidate its implications in the aeroelastic divergence problem of composite swept wings.

The aeroelastic tailoring concept applied to composite lifting surfaces in general and to forward-swept wings especially was discussed recently and thoroughly analyzed in a series of excellent survey papers in which the state-of-the-art of the problem was presented.[22-24]

Basic Assumptions

Consider the case of a swept (back or forward) wing structure, idealized as a box beam whose upper and lower faces are constructed of laminated composite materials. As in Refs. 5 and 6, we shall postulate also the existence of a reference axis (RA), coinciding with the y axis and located in the reference plane of the box beam and at mid-distance between its front and rear edges.

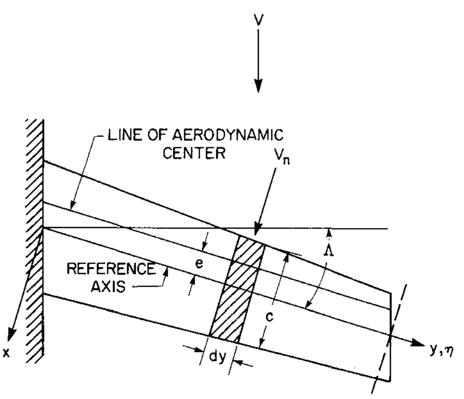

Fig. 1 Geometry of the swept wing aircraft.

The angle of sweep (considered positive for swept-back wings and negative for swept-forward wings) is measured in the x-y plane of the wing from the direction normal to the airflow to the reference axis. The wing is considered clamped normal to this RA, its effective length ℓ being measured along this axis. All parameters associated with the wing sections, such as chord, location of the aerodynamic center, etc., are based on sections normal to the RA (see Fig. 1).

The material of each constituent lamina is assumed to be orthotropic. Without contrary mentions, the orthotropicity angle $\theta_{(j)}$ of each lamina is measured in the counterclockwise direction starting from the rearward normal to the y axis. In addition to these assumptions, we shall postulate that both the chordwise deformation and wing distortions are negligibly small.

Governing Equations

For the sake of completeness and definition of the involved quantities, a short derivation of the equations governing the aeroelastic equilibrium of a composite wing structure is presented in the Appendix. In their deduction, the above-mentioned assumptions are appropriately used. For the static case considered in this paper, the aerodynamic terms L and T intervening in the equilibrium equations and representing the lift and the aerodynamic torsional moment (per unit length), respectively, are expressed as

$$L(y) = q_n c a_0 \theta_{\text{eff}} \tag{1a}$$

$$T(y) = q_n c e a_0 \theta_{\text{eff}} \tag{1b}$$

where $q_n (\rho_0/2) V_n^2 = q \cos^2\Lambda$ denotes the dynamic pressure component normal to the leading edge; a_0 [$\equiv 2\pi \text{AR}/(\text{AR}+4\cos\Lambda)$] the lift curve slope coefficient, where AR is the wing aspect ratio and e the distance between the spanwise reference axis and the line of aerodynamic centers (for additional refinements incorporating compressibility effects, see Ref. 25); $\theta_{\text{eff}} = \theta - Z_0' \tan\Lambda$, where θ denotes the twist about the AR and $Z_0'(\equiv \mathrm{d}Z_0/\mathrm{d}y)$ the bending slope of the AR measured along this axis. (The twisting angle θ is not to be confused with the arthotropicity angle θ_j). With all of these in view, the equations governing the static aeroelastic equilibrium of nonuniform composite swept wings reduce to

$$(\tilde{\mathfrak{D}}_{22} Z_0'')'' - (\tilde{\mathfrak{D}}_{26} \theta')'' = q c a_0 \cos^2\Lambda (\theta - Z_0' \tan\Lambda) \tag{2a}$$

$$\left(\frac{c^3}{12} D_{22} \theta''\right)'' - (\tilde{\mathfrak{D}}_{66}\theta')' + (\tilde{\mathfrak{D}}_{26} Z_0'')' = q c e a_0 \cos^2\Lambda (\theta - Z_0' \tan\Lambda) \tag{2b}$$

These equations are to be supplemented by the appropriate boundary conditions [Eqs. (A14) and (A15)].

In the following, Eqs. (2) will be converted to an integral form having an energetic meaning. Toward this end, Eqs. (2a) and (2b) will be multiplied by $Z_0 \mathrm{d}y$ and $\theta \mathrm{d}y$, respectively, and integrated over $[0,\ell]$. (In the terminology of the functional analysis, this operation is referred to as the scalar multiplication.)[26] Introduction of the nondimensional deflection $\tilde{Z}(\equiv Z_0/\ell)$ and spanwise coordinate $\eta(\equiv y/\ell)$, followed by the partial integration of the obtained equations, whenever possible, results in the equilibrium equations expressed in integral form as well as in some terms to be evaluated at $\eta = 0, 1$, which vanish by virtue of the boundary conditions [Eqs. (A15) and (A16)]. In this modified form, the equations are

$$\int_0^1 \tilde{\mathfrak{D}}_{22} (\tilde{Z}_{,11})^2 \mathrm{d}\eta - \int_0^1 \tilde{\mathfrak{D}}_{26} \tilde{Z}_{,11} \theta_{,1} \mathrm{d}\eta - q_n a_0 \ell^3 \left[\int_0^1 c\theta \tilde{Z} \mathrm{d}\eta - \tan\Lambda \int_0^1 c\tilde{Z}\tilde{Z}_{,1} \mathrm{d}\eta\right] = 0 \tag{3a}$$

$$(12\ell^2)^{-1}\int_0^1 D_{22}c^3(\theta_{,11})^2 d\eta + \int_0^1 \tilde{\mathfrak{D}}_{66}(\theta_{,1})^2 d\eta$$

$$-\int_0^1 \tilde{\mathfrak{D}}_{26}\tilde{Z}_{,11}\theta_{,1}d\eta - q_n a_0 \ell^2\left[\int_0^1 ce\theta^2 d\eta\right.$$

$$\left. -\tan\Lambda\int_0^1 ce\theta\tilde{Z}_{,1}d\eta\right] = 0 \tag{3b}$$

where $\partial(\ )/\partial\eta \equiv (\ )_{,1}$.

Equations (3) express a balance of energies, with each term playing a certain role. The lowest positive value of the dynamic pressure q, for which the elastic stored energies equal those furnished by the airstream, corresponds to the wing divergence speed. In other words, the minimum value of q, for which the balance of energies as governed by Eqs. (3) vanishes identically, corresponds to the divergence instability conditions. The above equations may be used as well for approaching the static instability of both variable and uniform composite swept wings.

Case of Geometrically Similar Cross-Sectional Wings

In this case, the geometrical and structural parameters of the composite wing may be expressed as (see, e.g., Refs. 1 and 6)

$$c(\eta) = \xi c_R, \qquad e(\eta) = \xi e_R$$

$$\tilde{\mathfrak{D}}_{22}(\eta) = \xi^4\tilde{\mathfrak{D}}_{22}^R, \qquad \tilde{\mathfrak{D}}_{66}(\eta) = \xi^4\tilde{\mathfrak{D}}_{66}^R$$

$$\tilde{\mathfrak{D}}_{26}(\eta) = \xi^4\tilde{\mathfrak{D}}_{26}^R, \qquad D_{22}(\eta) = \xi^4 D_{22}^R \tag{4}$$

where $\xi = 1-(1-\sigma)\eta$ and $\sigma(\equiv c_T/c_R)$ is the wing taper ratio $(0<\sigma\leq 1)$. Here, the superscript (or subscript) R affecting a certain quantity identifies its affiliation to the wing root section.

By virtue of Eqs. (4), Eqs. (3) may be reduced to

$$\int_0^1 \xi^4(\tilde{Z}_{,11})^2 d\eta - K\int_0^1 \xi^4\tilde{Z}_{,11}\theta_{,1}d\eta$$

$$-a_1\left[\int_0^1 \xi\theta\tilde{Z}d\eta - \tan\Lambda\int_0^1 \xi\tilde{Z}\tilde{Z}_{,1}d\eta\right] = 0 \tag{5a}$$

$$S_R\int_0^1 \xi^7(\theta_{,11})^2 d\eta + \int_0^1 \xi^4(\theta_{,1})^2 d\eta$$

$$-G\int_0^1 \xi^4\tilde{Z}_{,11}\theta_{,1}d\eta - a_2\left[\int_0^1 \xi^2\theta^2 d\eta\right.$$

$$\left. -\tan\Lambda\int_0^1 \xi^2\theta\tilde{Z}_{,1}d\eta\right] = 0 \tag{5b}$$

where

$$K \equiv \tilde{\mathfrak{D}}_{26}^R/\tilde{\mathfrak{D}}_{22}^R, \qquad G \equiv \tilde{\mathfrak{D}}_{26}^R/\tilde{\mathfrak{D}}_{66}^R \tag{6a,b}$$

define the nondimensional structural bending-torsion coupling parameters (shown in Ref. 4 to satisfy the condition $KG<1$, where KG defines the cross-coupling stiffness parameter);

$$S_R \equiv \frac{c_r^3}{12\ell^2}\frac{D_{22}^R}{\tilde{\mathfrak{D}}_{66}^R} \tag{7}$$

is the warping stiffness parameter; and

$$a_1 = q_n c_r \ell^3 a_0/\tilde{\mathfrak{D}}_{22}^R \tag{8a}$$

$$a_2 = q_n e_r c_r \ell^2 a_0/\tilde{\mathfrak{D}}_{66}^R \tag{8b}$$

define two nondimensional speed parameters associated with the bending and torsional degrees of freedom, respectively.

$\theta(\eta)$ and $\tilde{Z}(\eta)$ will be expressed in the form[27]

$$\theta(\eta) = \zeta_\theta f(\eta) \tag{9a}$$

$$\tilde{Z}(\eta) = \zeta_Z W(\eta) \tag{9b}$$

where $f(\eta)$ and $W(\eta)$ are the normalized torsional and bending deflection functions [assumed to satisfy the boundary conditions of Eqs. (A14) and (A15)] and ζ_θ and ζ_Z the generalized coordinates. Inserting Eqs. (9) in Eqs. (5) and invoking the standard requirement of nontriviality for the solution of the obtained homogeneous system of equations results in the divergence condition

$$a_1^2 L_{11} + a_1 L_{12} + L_{13} = 0 \tag{10}$$

Its lowest positive root corresponds to the divergence speed of swept composite-type wings.

In Eq. (10), coefficients L_{ij} depend on the structural and geometrical characteristics of the composite wing and on the selected mode shape W and f as follows:

$$L_{11} = A\tan\Lambda\left[\int_0^1 \xi fW d\eta\int_0^1 \xi^2 fW_{,1}d\eta\right.$$

$$\left. -\int_0^1 \xi^2 f^2 d\eta\int_0^1 \xi WW_{,1}d\eta\right] \tag{11a}$$

$$L_{12} = S_R\tan\Lambda\int_0^1 \xi^7(f_{,11})^2 d\eta\int_0^1 \xi WW_{,1}d\eta$$

$$+\tan\Lambda\int_0^1 \xi^4(f_{,1})^2 d\eta\int_0^1 \xi WW_{,1}d\eta$$

$$-G\int_0^1 \xi^4 W_{,11}f_{,1}d\eta\int_0^1 \xi Wf d\eta$$

$$-A\left[\int_0^1 \xi^2 f^2 d\eta\int_0^1 \xi^4(W_{,11})^2 d\eta\right.$$

$$\left. -K\tan\Lambda\int_0^1 \xi^4 W_{,11}f_{,1}d\eta\int_0^1 \xi^2 fW_{,1}d\eta\right] \tag{11b}$$

$$L_{13} = S_R\int_0^1 \xi^7(f_{,11})^2 d\eta\int_0^l \xi^4(W_{,11})^2 d\eta$$

$$+\int_0^1 \xi^4(f_{,1})^2 d\eta\int_0^1 \xi^4(W_{,11})^2 d\eta$$

$$-KG\left[\int_0^1 \xi^4 W_{,11}f_{,1}d\eta\right]^2 \tag{11c}$$

In the deduction of Eqs. (11) use was made of the relationship

$$a_2 = a_1 A$$

where

$$A \equiv \frac{\tilde{\mathfrak{D}}_{22}^R e_R}{\ell\tilde{\mathfrak{D}}_{66}^R} = \left(\frac{e_r}{\ell}\frac{G}{K}\right) \tag{12}$$

Concerning the solution of Eq. (10), it should be stressed that its accuracy depends essentially on the appropriate selection of modal functions $f(\eta)$ and $W(\eta)$.

Special Cases of the General Equation

Several special cases will be considered next. They result by the appropriate specialization of the general equation (10). These cases are discussed in the following subsections.

Divergence of Swept Wings in Pure Bending

In this case, it may be assumed that the torsional rigidity is very large, allowing to consider $\tilde{D}_{66} \to \infty$. From Eq. (10), having in view Eqs. (11), (6), and (7), one obtains

$$(q_n)_D = -2\int_0^1 \tilde{D}_{22}(W_{,11})^2 d\eta \Big/ \ell^3 a_0 \tan\Lambda \int_0^1 c(W^2)_{,1} d\eta \quad (13)$$

This result, coinciding with the one obtained in Ref. 25, reveals that only a swept-forward wing ($\Lambda \to -\Lambda$) can diverge in bending. In this case, for reasons of aeroelastic optimization, the bending rigidity $\tilde{D}_{22}$ is to be increased as much as possible. In this connection, as may be inferred from Eqs. (A16), a symmetric laminate (for which the coupling rigidities $K_{ij} \to 0$) is preferable to an asymmetric one (for which $K_{ij} \neq 0$). For a uniform wing, by using the representation $W(\eta) = (6\eta^2 - 4\eta^3 + \eta^4)/3$ for $W(\eta)$, one readily obtains $\lambda_D = 6.40\ \tilde{D}_{22}/c\ell^3$. The difference with respect to the exact one (whose coefficient is 6.33) is only 1%. Here $\lambda \equiv q_n a_0 \tan\Lambda$, where Λ is considered a negative angle. It may be remarked that, in the pure bending case, the warping restraint effect becomes immaterial, as is well evident.

Divergence in Pure Torsion

In this case, it is assumed that bending rigidity $\tilde{D}_{22}$ is a large quantity, while the warping rigidity parameter remains a finite quantity. Conversion of Eq. (10) in terms of a_2, with the help of Eqs. (8) and employment of Eqs. (6), results in the following expression for $(q_n)_D$:

$$(q_n)_D = \frac{1}{\ell^2 a_0}\int_0^1 \tilde{D}_{66}(f_{,1})^2 d\eta \Big/ \int_0^1 cef^2 d\eta$$

$$\times \left[1 + \int_0^1 D_{22}c^3 (f_{,11})^2 d\eta \Big/ 12\ell^2 \int_0^1 \tilde{D}_{66}(f_{,1})^2 d\eta\right] \quad (14)$$

The last term in the bracket identifies the contribution of the warping restraint effect. Equation (14) shows that divergence instability in pure torsion may occur for straight wings only. In fact, as can readily be seen, Eq. (14) may be obtained alternatively by considering ab initio $\Lambda = 0$ and $K = G = 0$ (implying $\tilde{D}_{26} = 0$)—in this case, $(q_n)_D \to q_D$.

In the conditions of the warping restraint, the normalized function $f(\eta)$ fulfilling the boundary conditions [Eqs. (A21)] reads

$$f = \left(\frac{16}{5}\frac{1+3S}{1+18S} - \frac{1}{5}\right)\eta^4 - \frac{8(1+3S)}{1+18S}\eta^3 + \left[\frac{6}{5} + \frac{24(1+3S)}{5(1+18S)}\right]\eta^2 \quad (15)$$

where, for the case of a symmetric laminate

$$S_R \to S \equiv \frac{1}{48}\left(\frac{c}{\ell}\right)^2 \frac{D_{22}}{D_{66}} \quad (16)$$

In the limiting case $S \to 0$, by virtue of the appropriate boundary conditions [Eqs. (A21)], the normalized function $f(\eta)$ should be considered as

$$f(\eta) = 2\eta - \eta^2 \quad (17)$$

Having in view that for $S \to 0$, in conjunction with Eqs. (14) and (17), the solution to the divergence problem is $(a_2)_D = 2.5$ (instead of 2.47, the exact one), we can obtain on the basis of Eqs. (14-16) the variation of $(a_2^*/a_2)_D$ vs S, where $(a_2^*)_D$ and $(a_2)_D$ denote the divergence velocity parameter [Eq. (8b)], determined in the conditions of the warping restraint and of free warping, respectively. This variation is displayed in Fig. 2, which reveals that the warping restraint effect has a stabilizing influence in the pure torsion case. A similar trend was obtained previously for metallic wings in Ref. 16.

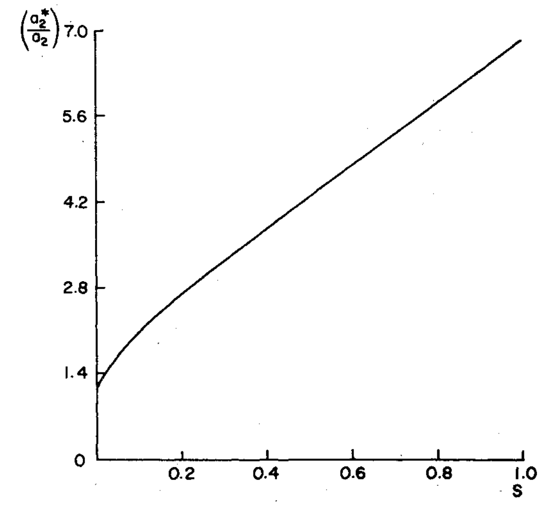

Fig. 2 Variation of the divergence velocity ratio $(a_2^*/a_2)_D$ with S.

Case of Uniform Swept Wings

Equation (10) of the divergence instability remains valid in this case. Appropriate to this case, its coefficients L_{ij} are obtained by specializing Eqs. (11) for $\xi \to 1$.

In such a manner, the properties of the uniform wing will correspond to the ones defined at the wing root. Having this fact in mind, in the following the index R will be suppressed.

The two distinct cases, corresponding to the inclusion ($S \neq 0$) and neglection ($S = 0$) of WRE, will be considered separately. Associated to these two cases, the boundary conditions at $\eta = 0,1$ are four and three in number, respectively [see Eqs. (A14), (A15), (A17), and (A18)]. The modal functions appropriate to these cases will be considered in the form for the free warping case,

$$f(\eta) = 3\eta - 3\eta^2 + \eta^3$$

$$W(\eta) = (6\eta^2 - 4\eta^3 + \eta^4)/3 \quad (18)$$

and for the restrained warping case

$$f(\eta) = 10\eta^2 - 20\eta^3 + 15\eta^4 - 4\eta^5$$

$$W(\eta) = (6\eta^2 - 4\eta^3 + \eta^4)/3 \quad (19)$$

Employment in Eqs. (12) and (13), specialized for $\xi \to 1$, of Eqs. (18) and (19) for f and W, results in the following approximate closed-form solutions of $q(\equiv q_D)$ for the free warping case:

$$q_D = \frac{2.8(1-KG)}{a_0 c\ell^3 \cos^2\Lambda} \times \left\{\tilde{D}_{22} \Big/ \left[\frac{1 - K\tan\Lambda}{(\ell/e)(\tilde{D}_{66}/\tilde{D}_{22})} - 0.4375(\tan\Lambda - G)\right]\right\} \quad (20)$$

and for the restrained warping case:

$$q_D = \frac{2.392(1.4 - KG) + 72.333S}{a_0 c\ell^3 \cos^2\Lambda} \times \left\{\tilde{D}_{22} \Big/ \left[\frac{1.19347 - K\tan\Lambda}{(\ell/e)(\tilde{D}_{66}/\tilde{D}_{22})} - 0.45712(1.144\tan\Lambda - G) - 1.3S\tan\Lambda\right]\right\} \quad (21)$$

Equation (20) is similar to Eq. (27) in Ref. 6 upon discarding the warping restraint effect. In spite of the slight differences in the coefficients (2.8 vs 2.47 and 0.4375 vs 0.39—the second quantities belonging to Weisshaar's expression for q_D), the numerical applications reveal small differences in the values of q_D, those obtainable from Ref. 6 being slightly more conservative than the present ones. For the case of composite unswept wings ($\Lambda \to 0$), when only the structural coupling is present, the employment of

$$f(\eta)=2\eta-\eta^2, \qquad W(\eta)=(3\eta^2-\eta^3)/2$$

results in a reduction in the differences of the two coefficients (i.e., 2.5 vs 2.47 and 0.4166 vs 0.39) and implicitly of the predicted critical values of q. It may be concluded that the present analysis may furnish rather accurate solutions when compared with the available exact ones (derived for the free warping case).

In addition, the present approach can provide quantitative results concerning the influence of WRE on the divergence instability characteristics of composite swept wings. In this respect, it should be pointed out that incorporation of WRE generates a series of qualitative and quantative differences with respect to the free warping counterpart. While the qualitative differences include the difference in the order of the equations governing instability in the divergence of composite swept wings and, as a result, in the difference of the associated number of boundary conditions to be fulfilled, the quantitative ones are of no less importance. These differences involving the values of the divergence speed can be of the order of 20–30%. In this context, it should be emphasized that: 1) in contrast to the case of metallic unswept wings where the stabilizing influence of the warping restraint effects for small-aspect-ratio wings declines for large or moderate AR wings, (see Ref. 16), in the case of composite swept wings, its influence may be strong also in the case of large AR wings; and 2) in contrast to the case of metallic swept wings in which the warping restraint effect has solely a stabilizing character, in the case of composite wings its influence can be also destabilizing.

In the same context, the concept of the divergence-free swept angle prompted by Weisshaar[5,6] can be useful in this case as well. The angle Λ_{cr} is defined as the minimum (positive) and maximum (negative) sweep angle above and below which, respectively, the divergence instability becomes impossible. The expressions in each case are obtained from Eqs. (20) and (21), resulting in

$$\tan\Lambda_{cr}=\frac{G+2.286(e/\ell)(G/K)}{1+2.286(e/\ell)G} \tag{22}$$

for the free warping case and

$$\tan\Lambda_{cr}=\frac{G+2.6108(e/\ell)(G/K)}{1.144+2.1876(e/\ell)G+24.72S} \tag{23}$$

for the restrained warping case.

In addition to the conclusion substantiated in Refs. 5 and 6 [and which could be reobtained from Eqs. (22) and (23)], according to which, in the case of swept composite wings, it is possible to determine such sweep angles in the forward-swept range for which the divergence instability is precluded at any flight speed, another conclusion may be established. From Eq. (23), it can be concluded that WRE may contribute to either the increasing or decreasing of Λ_{cr}. However, as seen from Eq. (23), this influence can be controlled through appropriate tailoring.

Numerical Illustrations and Conclusions

The accuracy of the present approach will be tested by comparing the results obtainable on the basis of Eq. (20) with their counterparts derived in Refs. 4 and 6 for a free warping model. Toward this goal, we shall consider the wing configuration analyzed in Refs. 4 and 6. It consists of $N=20$ plies of the same thickness ($t=0.12$ in.) and the same material (boron-epoxy). Its elastic constants are: $E_1=32.5\times10^6$ psi, $E_2=3.2\times10^6$ psi, $G_{12}=1.05\times10^6$ psi, and $\nu_{12}=0.36$. It is assumed that the angle of the ply orientation $\theta_{(j)}$ is equal for all the constituent layers and also that the wing is of constant chord, with AR $=6$, $e=0.1c$, and $a_0=5$.

Table 1 displays several values of $(a_1)_D$ obtained on the basis of the present approach [Eq. (20)] and the one developed in Refs. 4 and 6. These values are identified as $(a_1)_D$ and $(\tilde{a}_1)_D$, respectively. In Fig. 3, the variation of the ratio $(q_n^*/q_n)_D$ vs AR is depicted, where $(q_n^*)_D$ and $(q_n)_D$ denote the divergence dynamic pressure obtained by incorporating WRE [Eq. (21)] and by disregarding it [Eq. (20)], respectively.

The configuration as well as the structural characteristics, previously considered, are adopted in these instances. Figure 3 reveals the stabilizing influence of WRE, which, in contrast to the conventional metallic wings, is also present in the case of high AR wings. Figure 4 displays the variation of Λ_{cr} vs θ obtained for the cases of free warping [on the basis of Eq. (22) and its counterpart derived in Refs. 4 and 6] and of warping restraint [Eq. (23)]. For the free warping case, Fig. 4 reveals the coincidence of the present results with the ones obtained in Refs. 4 and 6. In addition, when the warping restraint effect is involved, this figure shows a notable modification of the minimum (positive) and maximum (negative) sweep angles at which the divergence instability ceases to exist.

However, as may be shown, the WRE considered in the context of the composite-type wings can be also detrimental. This is shown in the case of a wing structure in which the plies of a graphite-epoxy material are arranged in the sequence $(90\text{ deg}/-45\text{ deg}/45\text{ deg}/0)_s$.

The graphite-epoxy material is characterized by the constants $E_1=30\times10^6$ psi, $E_2=0.7\times10^6$ psi, $G_{12}=0.375\times10^6$ psi, $\nu_{12}=0.25$, and $t=0.005$ in. resulting in the nondimensional coupling parameters $K=-0.13296$ and $G=-0.20782$. By considering the remaining wing parameters to be the same as in the previous example, based on Eqs. (20) and (21), we obtain the variation of $(q_n^*/q_n)_D$ vs AR displayed in Fig. 5, which reveals the trend mentioned above.

These numerical findings enforce the conclusion that in the case of composite wings the warping restraint effect is to be taken into consideration whenever their divergence instability is investigated.

Appendix

The wing structure is idealized as a laminated composite flat plate whose constituent laminas are characterized by different orthotropicity angles and different material and thickness properties. The interface plane between the contiguous layers r and $(r+1)$ $(1<r<N$, where N denotes the total number of constituent layers) will be selected as the reference plane of the composite structure.

The points of the reference plane (defined by $z=0$) will be referred to a Cartesian system of coordinates x,y (see Fig. 1), where the upward z coordinate is considered perpendicular to the (x,y) plane. Adopting the Kirchhoff assumptions for the composite plate as a whole, which yields

$$V_1=v_1-z\frac{\partial V_3}{\partial x_1}, \quad V_2=v_2-z\frac{\partial V_3}{\partial x_2}, \quad V_3=v_3 \tag{A1}$$

where

$$V_i\equiv V_i(x,y,z), \quad v_i\equiv v_i(x,y)=V_i\,|_{z=0}$$

Representing the deflection v_3 in the form,

$$v_3(x,y)=Z_0(y)-x\theta(y) \tag{A2}$$

Table 1 Comparison of present $[(a_1)_D]$ and Weisshaar's $[(\tilde{a}_1)_D]$ results

	$\theta=0$ deg $K=0$ $G=0$	$\theta=30$ deg $K=1.26066$ $G=0.24362$	$\theta=60$ deg $K=0.96594$ $G=0.7292$	$\theta=90$ deg $K=0$ $G=0$	$\theta=120$ deg $K=-1.06594$ $G=-0.7292$	$\theta=150$ deg $K=-1.26066$ $G=-0.24362$	$\theta=180$ deg $K=0$ $G=0$
				$\Lambda=0$ deg			
$(a_1)_D$	97.174	17.1645	2.4050	9.569	No	No	97.1746
$(\tilde{a}_1)_D$	96.008	16.868	2.3587	9.454	divergence	divergence	96.009
				$\Lambda=-30$ deg			
$(a_1)_D$	10.060.	5.2391	1.355	5.449	No	13.130	10.060
$(\tilde{a}_1)_D$	9.8449	5.1654	1.331	5.078	divergence	12.9051	9.845
				$\Lambda=-60$ deg			
$(a_1)_D$	3.574	2.1924	0.724	2.748	1.962	3.015	3.574
$(\tilde{a}_1)_D$	3.5223	2.1635	0.711	2.637	1.951	2.988	3.522

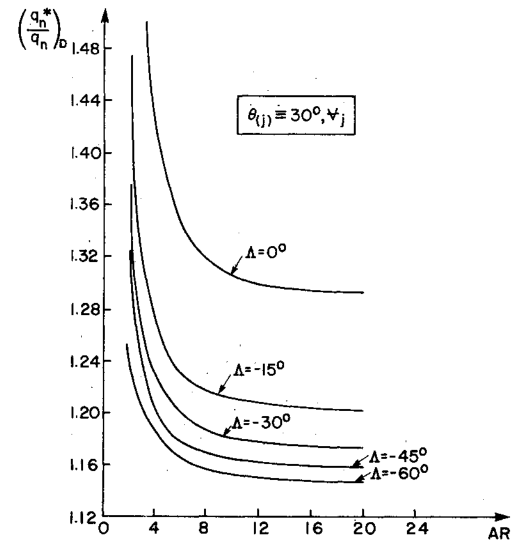

Fig. 3 Ratio of $(q_n^*)_D$ with WRE included and $(q_n)_D$ determined for free warping as a function of AR for various sweep angles and for a laminate characterized by $\theta_{(j)}=30$ deg.

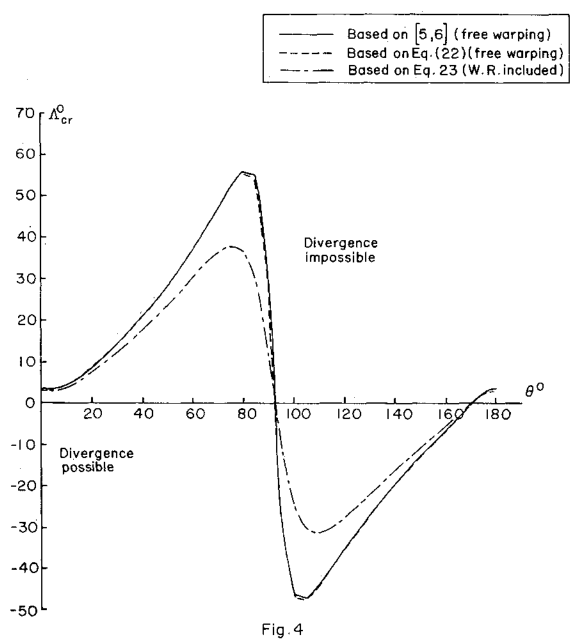

Fig. 4 Variation of the critical sweep angle Λ_{cr} with fiber orientation angle θ for free warping and warping restraint.

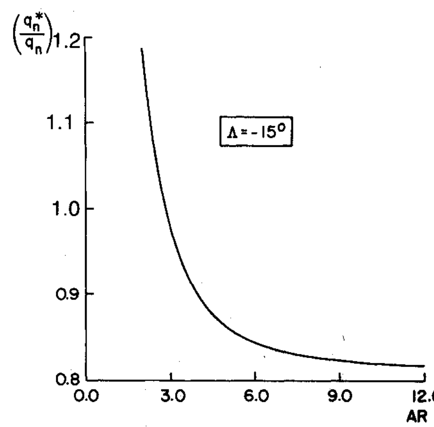

Fig. 5 Ratio of $(q_n^*)_D$ with WRE included and $(q_n)_D$ determined for free warping as a function of AR for a (90 deg/−45 deg/45 deg/0)$_s$ graphite-epoxy composite wing of $\Lambda=-15$ deg.

where $Z_0\equiv Z_0(y)$ and $\theta\equiv\theta(y)$ denote the deflection associated with the reference axis points and the twist around this axis, respectively, and postulating that the chordwise sections of the wing are rigid, which involves

$$v_1\to 0 \quad\text{and}\quad v_2\to v_2(y)$$

result in the following expressions for the two-dimensional strain components:

$$\epsilon_{11}=0,\quad \epsilon_{12}=0,\quad \epsilon_{22}=v_2'$$

$$\kappa_{11}=0,\quad \kappa_{12}=2\theta',\quad \kappa_{22}=-(Z_0''-x\theta'') \tag{A3}$$

where $\partial(\ )/\partial y\equiv(\ )'$ and ϵ_{ij} and κ_{ij} denote the stretching and bending strain components, respectively. In order to obtain the equations governing the static equilibrium of the composite wing and the associated boundary conditions (BC), the virtual work principle of the three-dimensional elasticity theory will be applied. Toward this end, the stationary condition of the functional

$$J_{3\text{-D}}\equiv U-\int_{\Omega_T} T_iV_i\,\mathrm{d}\Omega \tag{A4}$$

which is to be converted in terms of quantities depending on the y coordinate only, is to be determined. In Eq. (A4), U denotes the strain energy of the three-dimensional body, while T_i denotes the external loads applied on $\Omega_T\epsilon\Omega$. Selecting the reference axis located in the reference plane at the mid-distance between its front and rear edges, performing the integration in the expression of U across the thickness of the composite plate and in the chordwise direction, and defining the anisotropic stiffness quantities results in

$$C_{ij} = \sum_{k=1}^{N} (\bar{Q}_{ij})_{(k)} (h_{(k)} - h_{(k-1)}) \tag{A5a}$$

$$K_{ij} = \frac{1}{2}\left[\sum_{k=1}^{r} (\bar{Q}_{ij})_{(k)} (h^2_{(k)} - h^2_{(k-1)}) - \sum_{k=r+1}^{N} (\bar{Q}_{ij})_{(k)} (h^2_{(k)} - h^2_{(k-1)}) \right] \tag{A5b}$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{N} (\bar{Q}_{ij})_{(k)} (h^3_{(k)} - h^3_{(k-1)}) \qquad (i,j = 1,2,6) \tag{A5c}$$

which correspond to stretching, bending-stretching coupling, and bending rigidities, respectively (see Ref. 14). In Eqs. (A5), $h_{(k)}$ is the distance along the z coordinate from the reference plane to the top face of the kth layer and $(\bar{Q}_{ij})_{(k)}$ the reduced orthotropic moduli of the kth layer referred to the off-axis system represented by the (x,y) coordinates. With all these in view, the one-dimensional counterpart of Eq. (A4) becomes

$$J_{\text{1-D}} = U_C + U_K + U_D - \int_0^{\ell} (Yv_2 + LZ_0 + T\theta)\,\mathrm{d}y \tag{A6}$$

where

$$U_C \equiv \frac{1}{2}\int_0^{\ell} cC_{22}(v_2')^2 \mathrm{d}y \tag{A7a}$$

$$U_K \equiv \int_0^{\ell} c(2K_{26}v_2'\theta' - K_{22}v'Z_0'')\mathrm{d}y \tag{A7b}$$

$$U_D \equiv \frac{1}{2}\int_0^{\ell} (c[D_{22}(Z_0'')^2 + 4D_{66}(\theta')^2 - 4D_{26}\theta' Z_0'')\mathrm{d}y \tag{A7c}$$

denote the strain energies associated with the stretching, bending-stretching coupling, and bending, respectively, while Y, L, and T denote the spanwise, load, lift, and torsional aerodynamic moment, respectively.

From the stationary condition,

$$\delta J_{1-\mathrm{D}} = 0 \tag{A8}$$

by taking the variation of the involved quantities, by performing the integration by parts whenever possible, by collecting terms, and by setting the coefficients of δZ_0, $\delta\theta$, and δv_2 in Eq. (A6) [considered in conjunction with Eq. (A7)], to be zero, one obtains: 1) the equations governing the static equilibrium of nonuniform composite wings given by

$$\delta v_2: \quad (cC_{22}v_2' - cK_{22}Z_0'' + 2cK_{26}\theta')' = -Y \tag{A9a}$$

$$\delta\theta: \quad \left(\frac{c^3}{12}D_{22}\theta''\right)'' - 4(cD_{66}\theta')' + 2(cD_{26}Z_0'')' - 2(cK_{26}v_2')' - T = 0 \tag{A9b}$$

$$\delta Z_0: \quad (cD_{22}Z_0'')'' - 2(cD_{26}\theta')'' - (cK_{22}v_2')'' = L \tag{A9c}$$

and 2) the consistent BC given by

$$v_2 = Z_0 = Z_0' = \theta = \theta' = 0 \quad \text{at} \quad y = 0 \tag{A10}$$

$$cC_{22}v_2' - cK_{22}Z_0'' + 2cK_{26}\theta' = 0 \tag{A11a}$$

$$cD_{22}Z_0'' - 2cD_{26}\theta' - cK_{22}v_2' = 0 \tag{A11b}$$

$$(cD_{22}Z_0'' - 2cD_{26}\theta' - cK_{22}v_2')' = 0 \tag{A11c}$$

$$-(c^3D_{22}\theta''/12)' + 4cD_{66}\theta' - 2cD_{26}Z_0'' + 2cK_{26}v_2' = 0 \tag{A11d}$$

$$c^3D_{22}\theta''/12 = 0 \quad \text{at} \quad y = \ell \tag{A11e}$$

However, it may easily be seen that in the absence of the spanwise load Y, the tenth-order governing equation system expressed in terms of v_2, Z_0, and θ may exactly be reduced to an eighth-order governing equation system expressed in terms of θ and Z_0 only. Toward this end, integration of Eq. (A9a), provided $Y=0$ and comparison with Eq. (A11a) yields:

$$cC_{22}v_2' - cK_{22}Z_0'' + 2cK_{26}\theta' = 0 \tag{A12}$$

Substitution of v_2' as it results from Eq. (A12) in Eqs. (A9b–A9c) yields the appropriate governing equations,

$$(\tilde{\mathfrak{D}}_{22}Z_0'' - \tilde{\mathfrak{D}}_{26}\theta')'' = L$$

$$(c^3D_{22}\theta''/12)'' - (\tilde{\mathfrak{D}}_{66}\theta' - \tilde{\mathfrak{D}}_{26}Z_0'')' = T \tag{A13}$$

In a similar way, the BC equations (A10) and (A11) modify as

$$Z_0 = Z_0' = \theta = \theta' = 0 \quad \text{at} \quad y = 0 \tag{A14}$$

$$\tilde{\mathfrak{D}}_{22}Z_0'' - \tilde{\mathfrak{D}}_{26}\theta' = 0$$

$$(\tilde{\mathfrak{D}}_{22}Z_0'' - \tilde{\mathfrak{D}}_{26}\theta')' = 0$$

$$-(c^3D_{22}\theta''/12)' + \tilde{\mathfrak{D}}_{66}\theta' - \tilde{\mathfrak{D}}_{26}Z_0'' = 0$$

$$D_{22}c^3\theta''/12 = 0 \quad \text{at} \quad y = \ell \tag{A15}$$

In Eqs. (A13) and (A15), $\tilde{\mathfrak{D}}_{22}$, $\tilde{\mathfrak{D}}_{26}$, and $\tilde{\mathfrak{D}}_{66}$ are the coupling stiffness parameters expressed by

$$\tilde{\mathfrak{D}}_{22} \equiv c(D_{22} - K_{22}^2/C_{22})$$

$$\tilde{\mathfrak{D}}_{26} \equiv 2c(D_{26} - K_{22}K_{26}/C_{22})$$

$$\tilde{\mathfrak{D}}_{66} \equiv 4c(D_{66} - K_{26}^2/C_{22}) \tag{A16}$$

As it may easily be observed, the term $c^3D_{22}\theta''''/12$ present in Eq. (A13) identifies the warping restraint effect. When this effect is ignored, from Eqs. (A10), (A11), and (A13) the pertinent governing equations result as

$$(\tilde{\mathfrak{D}}_{22}Z_0'' - \tilde{\mathfrak{D}}_{26}\theta')'' = L$$

$$(\tilde{\mathfrak{D}}_{66}\theta' - \tilde{\mathfrak{D}}_{26}Z_0'')' = -T \tag{A17}$$

while the associated BC read

$$Z_0 = Z_0' = \theta = 0 \quad \text{at} \quad y = 0 \tag{A18}$$

$$\tilde{\mathfrak{D}}_{22}Z_0'' - \tilde{\mathfrak{D}}_{26}\theta' = 0$$

$$(\tilde{\mathfrak{D}}_{22}Z_0'' - \tilde{\mathfrak{D}}_{26}\theta')' = 0$$

$$\tilde{\mathfrak{D}}_{66}\theta' - \tilde{\mathfrak{D}}_{26}Z_0'' = 0 \quad \text{at} \quad y = \ell \tag{A19}$$

Equations (A18) and (A19) are consistent with the sixth-order governing equation system [Eqs. (A17)]. Equations (A17–A19) coincide with the ones obtained in Ref. 4, while Eqs. (A13–A15) specialized for the case of uniform wings are similar to the ones presented in Ref. 12. For the case of unswept metallic type wings, Eqs. (A13–A15) reduce to those obtained in Ref. 16 by using the theory of thin-walled beams with warping inhibition.

For the sake of completeness, we also record here the equations governing the divergence in pure torsion of uniform wings with WRE. This case may be obtained from the previous general one by specializing the associated equations for $\Lambda \to 0$, $K = G \to 0$ (involving $\tilde{\mathfrak{D}}_{26} \to 0$). For a symmetric laminate, the pertinent equations, expressed in nondimensional variables, read

$$S\theta_{,1111} - \theta_{,11} = a_2\theta \tag{A20}$$

with the associated two boundary conditions at each edge

$$\theta = \theta_{,1} = 0 \text{ at } \eta = 0$$

$$\theta_{,11} = \theta_{,1} - S\theta_{,111} = 0 \text{ at } \eta = 1 \tag{A21}$$

while for the free warping case, it results

$$\theta_{,11} + a_2\theta = 0 \tag{A22}$$

with

$$\theta = 0 \text{ at } \eta = 0 \text{ and } \theta_{,1} = 0 \text{ at } \eta = 1$$

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